

Higher-order Wilson coefficients for $c \rightarrow u$ transitions in the standard model

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ABSTRACT: The standard theoretical framework to deal with weak decays of heavy mesons is the so-called weak effective Hamiltonian. It involves the short-distance Wilson coefficients, which depend on the renormalisation scale μ . For specific calculations one has to evolve the Wilson coefficients down from the electroweak scale $\mu = M_W$ to the typical mass scale of the decay under consideration. This is done by solving a renormalisation group equation for the effective operator basis. In this paper the results of a consistent two-step running of the $c \rightarrow u \ell^+ \ell^-$ Wilson coefficients for dimension-6 operators are presented. This running involves the intermediate scale $\mu = m_b$ (with $M_W > m_b > m_c$) where the bottom quark is integrated out. The matching coefficients and anomalous dimensions are taken to the required order by generalizing and extending results from $b \rightarrow s$ or $s \rightarrow d$ transitions available in the literature.

KEYWORDS: Effective field theories, Perturbative QCD, Renormalization Group

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1 Introduction

The study of flavour-changing neutral current (FCNC) transitions is a key tool to explore the generational structure of standard model (SM) fermions, and to look for physics beyond the standard model (BSM). A lot of work has been done to analyse processes involving b -quarks where in the meantime theoretical predictions and experimental measurements have reached a high level of precision [1]. In contrast to that, investigations of charm FCNCs are much less advanced due to several reasons. The corresponding rates are highly GIM-suppressed [2], experimental analyses are challenging, and decay modes are subjected to resonance contributions, shielding the electroweak physics. In many cases, extensions of the SM may upset the GIM suppression and give contributions which are sometimes orders of magnitude larger than within the SM.

Due to the specific CKM and mass structure of charm FCNCs, also the electroweak contributions within the SM can differ by several orders of magnitude depending on which corrections are taken into account [3]. It is therefore desirable to extend the SM calculation for the $c \rightarrow u\ell^+\ell^-$ transition to $\mathcal{O}(\alpha_s)$ within renormalisation-group improved perturbation theory. As a first step the weak effective Hamiltonian consisting of all relevant dimension-6 operators with the corresponding Wilson coefficients is needed to this order. In this paper we will present results for this step at next-to-next-to-leading logarithmic (NNLL) order which is required for a consistent treatment of the decays at $\mathcal{O}(\alpha_s)$.

The calculation of the Wilson coefficients is in many parts analogous to the one in the B -meson sector. The main difference is that in the case considered here, we have to perform a two-step matching. In addition to the matching at the high scale M_W , the bottom-threshold is crossed when evolving the renormalisation scale down to the charm mass. Therefore the bottom-quark has to be integrated out which leads to non-trivial matching conditions at the scale $\mu = m_b$. The running of the coefficients at the intermediate steps $M_W > \mu > m_b$ and $m_b > \mu$ can be performed analogously to the decay $b \rightarrow d/s\ell^+\ell^-$, where only the charge assignments and the number of active flavours of the corresponding

anomalous dimensions have to be adapted accordingly. The matching conditions at the high scale and the anomalous dimensions are known at the NNLL order [4–11].

In the next section we will present the effective Hamiltonian relevant for $c \rightarrow u$ transitions. The matching conditions at the high scale M_W and the relevant formulae for the running down to the charm scale are given. As some of the anomalous dimension matrices are only presented with explicit assignments for the quark charges and number of flavours for bottom decays in the literature, we will present them with the full parameter dependence. At the end of that section, the numerical values of the Wilson coefficients at the charm-mass scale are given and will be compared to the corresponding coefficients for b -decays. In section 3 we will focus on the clarification of some misunderstanding present in previous work. We will therefore present the effective Wilson coefficient C_9^{eff} at order α_s^0 and compare the results with existing treatments in the literature. Finally, in the appendix, we give formulae to switch between different operator bases for the effective weak Hamiltonian.

2 Effective Hamiltonian for $c \rightarrow u\ell\ell$

The short-distance expansion has to be divided into two steps: firstly, we integrate out the weak gauge bosons at a scale $\mu_W \sim M_W$. At this step, there are no penguin operators generated, as all d -type quark masses should be treated as massless [3] and the GIM mechanism is in full effect. The effective Hamiltonian for scales $M_W > \mu > m_b$ is given by

$$H_{\text{eff}}(M_W > \mu > m_b) = \frac{4G_F}{\sqrt{2}} \sum_{q=d,s,b} V_{cq}^* V_{uq} [C_1(\mu) \mathcal{O}_1^q + C_2(\mu) \mathcal{O}_2^q], \quad (2.1)$$

where

$$\mathcal{O}_1^q = (\bar{u}_L \gamma_\mu T^a q_L) (\bar{q}_L \gamma^\mu T^a c_L), \quad (2.2)$$

$$\mathcal{O}_2^q = (\bar{u}_L \gamma_\mu q_L) (\bar{q}_L \gamma^\mu c_L), \quad (2.3)$$

T^a are the generators of SU(3), and the subscript L denotes left-handed fields. Secondly, one integrates out the bottom-quark around $\mu_b \sim m_b$. This generates penguin operators with Wilson coefficients depending on M_W solely through $C_{1,2}(m_b)$. The effective Hamiltonian for scales $m_b > \mu > m_c$ is thus given by

$$H_{\text{eff}}(m_b > \mu > m_c) = \frac{4G_F}{\sqrt{2}} \sum_{q=d,s} V_{cq}^* V_{uq} [C_1(\mu) \mathcal{O}_1^q + C_2(\mu) \mathcal{O}_2^q + \sum_{i=3}^{10} C_i(\mu) \mathcal{O}_i], \quad (2.4)$$

with

$$\mathcal{O}_3 = (\bar{u}_L \gamma_\mu c_L) \sum_{q=u,d,s,c} (\bar{q} \gamma^\mu q), \quad (2.5)$$

$$\mathcal{O}_4 = (\bar{u}_L \gamma_\mu T^a c_L) \sum_{q=u,d,s,c} (\bar{q} \gamma^\mu T^a q), \quad (2.6)$$

$$\mathcal{O}_5 = (\bar{u}_L \gamma_\mu \gamma_\nu \gamma_\rho c_L) \sum_{q=u,d,s,c} (\bar{q} \gamma^\mu \gamma^\nu \gamma^\rho q), \quad (2.7)$$

$$\mathcal{O}_6 = (\bar{u}_L \gamma_\mu \gamma_\nu \gamma_\rho T^a c_L) \sum_{q=u,d,s,c} (\bar{q} \gamma^\mu \gamma^\nu \gamma^\rho T^a q), \quad (2.8)$$

$$\mathcal{O}_7 = -\frac{g_{\text{em}} m_c}{16\pi^2} (\bar{u}_L \sigma^{\mu\nu} c_R) F_{\mu\nu}, \quad (2.9)$$

$$\mathcal{O}_8 = -\frac{g_s m_c}{16\pi^2} (\bar{u}_L \sigma^{\mu\nu} T^a c_R) G_{\mu\nu}^a, \quad (2.10)$$

$$\mathcal{O}_9 = \frac{\alpha_{\text{em}}}{4\pi} (\bar{u}_L \gamma_\mu c_L) (\bar{\ell} \gamma^\mu \ell), \quad (2.11)$$

$$\mathcal{O}_{10} = \frac{\alpha_{\text{em}}}{4\pi} (\bar{u}_L \gamma_\mu c_L) (\bar{\ell} \gamma^\mu \gamma_5 \ell). \quad (2.12)$$

The sign convention for $\mathcal{O}_{7,8}$ corresponds to $+ig_s T^a$, $+ig_{\text{em}} e_f$ for the ordinary quark-gauge-boson vertex ($e_f = -1$ for charged lepton fields). Only $C_{1/2}$ receive non-zero contributions from the matching procedure at $\mu \sim M_W$, all Wilson coefficients of the penguin operators vanish identically as noted above. As a consequence C_{3-9} receive non-zero contributions only from the matching of the five-flavour effective theory above the scale m_b to the four-flavour effective theory below that scale and from the mixing of $\mathcal{O}_{1/2}$ into \mathcal{O}_{3-9} below the scale m_b , where the b -quark has been integrated out. C_{10} does not mix under renormalisation and thus is zero at all scales to leading order in the $1/M_W$ expansion.

Our aim is to determine the Wilson coefficients at a perturbative order which is suitable for performing analyses for D -decays¹ at first order in the strong coupling α_s . Because the anomalous dimension of \mathcal{O}_9 begins at order α_s^0 , the Wilson coefficient C_9 is needed to NNLL accuracy. This requires also the coefficients of the four-quark operators to this accuracy. At the scale $\mu \sim m_c$, the Wilson coefficients C_{1-8} are given by

$$C(\mu) = U^{(n_f=4)}(\mu, m_b) R U^{(n_f=5)}(m_b, M_W) C(M_W), \quad (2.13)$$

where $C(\mu)$ is to be understood as the vector of Wilson coefficients. In the following we will not present the results for the coefficients $C_{7/8}$, but rather for the renormalisation-scheme independent effective ones defined by

$$C_{7/8}^{\text{eff}}(\mu) = C_{7/8}(\mu) + \sum_{i=1}^6 y_i^{(7/8)} C_i(\mu), \quad (2.14)$$

with $y^{(7)} = Q(0, 0, 1, \frac{4}{3}, 20, \frac{80}{3})$ and $y^{(8)} = (0, 0, 1, -\frac{1}{6}, 20, -\frac{10}{3})$ in the chosen operator basis. One has to make the assignments $Q = Q_u = 2/3$ and $Q = Q_d = -1/3$ for D -decays and B -decays, respectively.

$U^{(n_f)}(\mu_1, \mu_2)$ is the evolution matrix which includes the renormalisation-group improved contributions from the scale μ_2 down to μ_1 and R is the matching matrix between the five- and four-flavour effective theory. As noted above, the vector containing C_{1-8} at

¹The results presented here can of course also be used for charmed baryons like Λ_c .

the scale M_W , $C(M_W)$, has only two non-zero entries, which are given by [8]

$$C_1(M_W) = 15a_s + a_s^2 \left[(16x + 8)\sqrt{4x - 1} \operatorname{Cl}_2 \left(2 \arcsin \frac{1}{2\sqrt{x}} \right) - \left(16x + \frac{20}{3} \right) \ln x - 32x + \frac{7091}{72} + \frac{17}{3}\pi^2 \right], \quad (2.15)$$

$$C_2(M_W) = 1 + a_s^2 \left(\frac{127}{18} + \frac{4}{3}\pi^2 \right), \quad (2.16)$$

where $x = [\hat{m}_t(M_W)/M_W]^2$ with the top quark $\overline{\text{MS}}$ -mass \hat{m}_t and $a_s = \alpha_s/(4\pi)$. The Clausen-function is defined as

$$\operatorname{Cl}_2(x) = \operatorname{Im} [\operatorname{Li}_2(e^{ix})], \quad (2.17)$$

with the dilogarithm Li_2 . The evolution matrix $U^{(n_f)}(\mu_1, \mu_2)$ satisfies

$$\frac{d}{d \ln \mu_1} U^{(n_f)}(\mu_1, \mu_2) = \gamma^T(n_f, \mu_1) U^{(n_f)}(\mu_1, \mu_2). \quad (2.18)$$

The solution for this matrix at NNLL order is given in (C.6) in [12] for B -decays. Trivial changes have to be incorporated for the case considered here. The anomalous dimension matrix is expanded as

$$\gamma(n_f, \mu_1) = \gamma^{(0)} a_s(n_f, \mu_1) + \gamma^{(1)} a_s(n_f, \mu_1)^2 + \dots \quad (2.19)$$

The 6×6 submatrix of the anomalous dimension with full n_f dependence can be found in [7, 10]. The 2×2 submatrix from self-mixing in the dipole operator sector is given in [11]. This matrix depends also on the charges of the quarks, which have to be chosen appropriately for the case of D -decays considered in this paper. Up to the required order, the 6×2 submatrix from mixing between four fermion and dipole operators has only been given in the literature for B -decays [9]. With the full dependence on the charges and active flavours it reads [13]

$$\gamma_{6 \times 2}^{(\text{eff}, 0)} = \begin{pmatrix} -\frac{4}{3}q_1 - \frac{8}{81}q_2 & \frac{173}{162} \\ 8q_1 + \frac{16}{27}q_2 & \frac{70}{27} \\ \frac{176}{27}q_2 & \frac{14}{27} \\ \left(-\frac{88}{81} + \frac{16}{27}n_f\right)q_2 & \frac{74}{81} - \frac{49}{54}n_f \\ \frac{6272}{27}q_2 & \frac{1736}{27} + 36n_f \\ 48n_1\bar{q} + \left(-\frac{3136}{81} + \frac{1456}{27}n_f\right)q_2 & \frac{2372}{81} + \frac{160}{27}n_f \end{pmatrix}, \quad (2.20)$$

$$\gamma_{6 \times 2}^{(\text{eff}, 1)} = \begin{pmatrix} \left(-\frac{374}{27} + \frac{2}{27}n_f\right)q_1 + \left(-\frac{12614}{729} + \frac{64}{729}n_f\right)q_2 & \frac{65867}{5832} + \frac{431}{5832}n_f \\ \left(\frac{136}{9} - \frac{4}{9}n_f\right)q_1 + \left(\frac{2332}{243} - \frac{128}{243}n_f\right)q_2 & \frac{10577}{486} - \frac{917}{972}n_f \\ -\frac{112}{3}n_1\bar{q} - \left(\frac{97876}{243} + \frac{4720}{243}n_f\right)q_2 & \frac{42524}{243} - \frac{2398}{243}n_f \\ -\frac{140}{9}n_1\bar{q} + \left(\frac{70376}{729} + \frac{4448}{729}n_f - \frac{32}{243}n_f^2\right)q_2 & -\frac{159718}{729} - \frac{39719}{5832}n_f - \frac{253}{486}n_f^2 \\ -\frac{3136}{3}n_1\bar{q} - \left(\frac{1764752}{243} + \frac{188608}{243}n_f\right)q_2 & \frac{2281576}{243} + \frac{140954}{243}n_f - 14n_f^2 \\ \gamma_{67}^{(\text{eff}, 1)} & -\frac{3031517}{729} - \frac{15431}{1458}n_f - \frac{6031}{486}n_f^2 \end{pmatrix}, \quad (2.21)$$

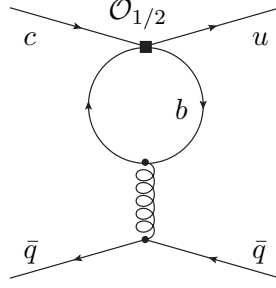


Figure 1. Diagrams relevant for the matching of the five-quark to the four-quark effective theory at order α_s .

with

$$\gamma_{67}^{(\text{eff},1)} = - \left(\frac{1136}{9} + \frac{56}{3} n_f \right) n_1 \bar{q} - \left(\frac{4193840}{729} - \frac{232112}{729} n_f + \frac{5432}{243} n_f^2 \right) q_2, \quad (2.22)$$

$$\bar{q} = q_1 - q_2.$$

For the case of D -meson decays one has to make the assignments $q_1 = Q_d = -1/3$, $q_2 = Q_u = 2/3$, $n_2 = 2$ and $n_1 = 3$ ($n_f = 5$) or $n_1 = 2$ ($n_f = 4$). The matrices given in the literature are reproduced with the following assignment for B -decays: $q_1 = Q_u = 2/3$, $q_2 = Q_d = -1/3$, $n_2 = 3$ and $n_1 = 2$.

The matrix R in (2.13) is the matching matrix from the five to the four active flavour effective theory. It is different from the unit matrix because the operators $\mathcal{O}_{1/2}$ are absent below the b -quark threshold. It is given by

$$R_{ij} = \delta_{ij} + a_s(m_b) R_{ij}^{(1)} + a_s(m_b)^2 R_{ij}^{(2)} + \dots \quad (2.23)$$

At order α_s the non-zero elements of $R_{ij}^{(1)}$ [14] are obtained from the diagrams depicted in figure 1 at zero momentum transfer:²

$$\begin{aligned} R_{41}^{(1)} &= -R_{42}^{(1)}/6 = 1/9, \\ R_{71}^{(1)} &= -R_{72}^{(1)}/6 = 8/81, \quad R_{81}^{(1)} = -R_{82}^{(1)}/6 = -1/54. \end{aligned} \quad (2.24)$$

The contributions at order α_s^2 are not known yet. The diagrams including an additional gluon connecting only the upper fermion lines in figure 1 have been calculated for B -physics in [15]. Unfortunately the calculation involves an expansion in m_c/m_b , which in the case considered here would turn into an expansion in m_b/m_c and is thus not applicable. In the following we will set $R^{(2)} \simeq 0$ as an approximation.

For C_9 we get the following evolution down to the scale $\mu \sim m_c$:

$$C_9(\mu) = C_9(m_b) + W^{(n_f=4)}(\mu, m_b) R U^{(n_f=5)}(m_b, M_W) C(M_W), \quad (2.25)$$

with the 1×6 matrix

$$W^{(n_f=4)}(\mu, m_b) = -\frac{1}{2} \int_{a_s(m_b)}^{a_s(\mu)} da_s \frac{\kappa(a_s)}{\beta(a_s)} U^{(n_f=4)}(\mu, m_b), \quad (2.26)$$

²Note that this is valid for the effective Wilson coefficients $C_{7/8}^{\text{eff}}$.

where $U^{(n_f=4)}(\mu, m_b)$ and R are the 6×6 submatrices from the corresponding quantities defined above. This time the vector $C(M_W)$ contains $C_1(M_W)$ to $C_6(M_W)$ where, as stated already, only two are non-vanishing. The solution of (2.26) can be found in (C.16) in [12]. The 1×6 matrix κ that describes the mixing into \mathcal{O}_9 is given by [9]

$$\kappa = \kappa^{(-1)} + \kappa^{(0)} a_s + \dots, \quad (2.27)$$

with

$$\kappa^{(-1)T} = \begin{pmatrix} -\frac{16}{9}q_1 \\ -\frac{4}{3}q_1 \\ -\frac{8}{3}q_2 - 8q \\ -\frac{32}{9}q_2 \\ -\frac{128}{3}q_2 - 80q \\ -\frac{512}{9}q_2 \end{pmatrix}, \quad (2.28)$$

$$\kappa^{(0)T} = \begin{pmatrix} -\frac{136}{27}q_1 - \frac{176}{243}q_2 \\ \frac{128}{9}q_1 + \frac{352}{81}q_2 \\ \frac{4160}{81}q_2 - 32q \\ \left(-\frac{784}{243} + \frac{544}{81}n_f\right)q_2 + \frac{64}{3}q \\ \frac{58112}{81}q_2 - 320q \\ \left(\frac{22784}{243} + \frac{4288}{81}n_f\right)q_2 + \frac{608}{3}q \end{pmatrix}, \quad (2.29)$$

$$\kappa^{(1)T} = \begin{pmatrix} -\frac{14999}{81}q_1 - \left(\frac{72560}{6561} + \frac{1120}{2187}n_f\right)q_2 + \frac{3152}{243}n_2\bar{q} + \frac{7976}{243}q \\ \frac{820}{27}q_1 + \left(\frac{333688}{2187} + \frac{2240}{729}n_f\right)q_2 - \frac{184}{81}n_2\bar{q} + \frac{1832}{81}q \\ \left(\frac{1524104}{2187} - \frac{44048}{729}n_f\right)q_2 - \left(\frac{2636}{9} - \frac{176}{9}n_f\right)q \\ \left(-\frac{1535926}{6561} + \frac{159620}{2187}n_f + \frac{608}{729}n_f^2\right)q_2 + \left(\frac{1201}{27} - \frac{32}{9}n_f\right)q \\ \left(\frac{31433600}{2187} + \frac{15904}{729}n_f\right)q_2 + \left(\frac{46552}{9} + \frac{2912}{9}n_f\right)q \\ \left(-\frac{48510784}{6561} + \frac{3516560}{2187}n_f + \frac{15872}{729}n_f^2\right)q_2 - \left(\frac{47624}{27} - \frac{1312}{27}n_f\right)q \end{pmatrix} \quad (2.30)$$

$$+ \zeta(3) \begin{pmatrix} \frac{352}{9}q_1 - \frac{640}{81}q_2 \\ -\frac{128}{3}q_1 + \frac{1280}{27}q_2 \\ \frac{256}{27}q_2 + 128q \\ \left(\frac{5056}{81} + \frac{1280}{27}n_f\right)q_2 + \frac{160}{3}q \\ \frac{4096}{27}q_2 + 1280q \\ \left(\frac{80896}{81} + \frac{12800}{27}n_f\right)q_2 - \frac{512}{3}q \end{pmatrix}, \quad (2.31)$$

where $q = n_1q_1 + n_2q_2$. The initial condition for C_9 at the scale m_b stems from the matching of the five-quark to the four-quark theory. The leading-order contribution arises from diagrams similar to the one in figure 1, but with the gluon exchanged by a photon and the quark-antiquark-pair by a lepton-pair. It is given by [14]

$$C_9(m_b) = -\frac{8}{27} \left(C_1(m_b) + \frac{3}{4}C_2(m_b) \right). \quad (2.32)$$

The two-loop contributions consist solely of diagrams like the ones calculated in [15]. Again, due to the expansion used there, we cannot use the results for our purpose and we will neglect these higher order contribution in our results.

	C_1	C_2	C_3	C_4	C_5	C_6
LL	−1.035	1.094	−0.004	−0.061	0.000	0.001
NLL	−0.712	1.038	−0.006	−0.093	0.000	0.001
NNLL	−0.633	1.034	−0.008	−0.093	0.000	0.001
	C_7^{eff}	C_8^{eff}	C_9	C_{10}	C_9^{NNLL}	C_{10}^{NNLL}
LL	0.078	−0.055	−0.098	0	−0.488	0
NLL	0.051	−0.062	−0.309	0		

Table 1. Wilson coefficients at the scale $\mu = 1.3 \text{ GeV}$ in leading-logarithmic (LL), next-to-leading-logarithmic (NLL) and next-to-next-to-leading-logarithmic (NNLL) order for C_{1-6} , C_9 and C_{10} . Input parameters are $\Lambda_{\overline{\text{MS}}}^{(4)} = 0.294 \text{ GeV}$, $\Lambda_{\overline{\text{MS}}}^{(5)} = 0.214 \text{ GeV}$, $\hat{m}_t(\hat{m}_t) = 163.3 \text{ GeV}$, $M_W = 80.4 \text{ GeV}$ and $\hat{m}_b(\hat{m}_b) = 4.18 \text{ GeV}$. 3-loop running is used for α_s .

We are now ready to present the results for the Wilson coefficients in table 1. It can be noted that the numerical results in the four-quark sector, C_{1-6} at the scale $\mu = 1.3 \text{ GeV}$, are not much different than the ones for b -decays at the scale $\mu = m_b$. Only C_1 is about twice as large for charm decays, whereas C_{2-6} are very similar. The main difference is observed for the coefficients C_7^{eff} , C_8^{eff} , C_9 and C_{10} . C_7^{eff} has a different sign and is roughly a factor of six smaller. C_8^{eff} is roughly a factor of three smaller than in b -decays. Whereas C_{10} is exactly zero to all orders in the strong coupling as explained above, also C_9 is an order of magnitude smaller for charm decays. Concerning the NNLL results, one has of course to bear in mind that we have neglected the two-loop matching conditions at the scale $\mu = m_b$.

One of us has already used the results presented here to perform a phenomenological analysis of D -decays [16].

3 Effective Wilson coefficient C_9^{eff}

We will now, analogously to the case of B -physics, introduce the renormalisation-scheme independent effective “Wilson coefficient” C_9^{eff} , which absorbs the universal long-distance effects from quark loops in perturbation theory [17]:

$$C_9^{\text{eff}}(\mu, s) = (V_{cd}^* V_{ud} + V_{cs}^* V_{us}) \left(C_9(\mu) + Y^{(ds)}(\mu, s) \right) + V_{cd}^* V_{ud} Y^{(d)}(\mu, s) + V_{cs}^* V_{us} Y^{(s)}(\mu, s), \quad (3.1)$$

where $s = q^2$, $q = p - p'$, with the momentum p and p' of the incoming c - and outgoing u -quark, respectively. The functions $Y^{(i)}(\mu, s)$ are defined as

$$Y^{(d)}(\mu, s) = h(\mu, s, 0) \left(\frac{4}{3} C_1(\mu) + C_2(\mu) \right), \quad (3.2)$$

$$Y^{(s)}(\mu, s) = h(\mu, s, m_s) \left(\frac{4}{3} C_1(\mu) + C_2(\mu) \right), \quad (3.3)$$

$$\begin{aligned}
Y^{(ds)}(\mu, s) = & -2h(\mu, s, m_c) \left(7C_3(\mu) + \frac{4}{3}C_4(\mu) + 76C_5(\mu) + \frac{64}{3}C_6(\mu) \right) \\
& + h(\mu, s, m_s) (6C_3(\mu) + 60C_5(\mu)) \\
& - \frac{4}{3}h(\mu, s, 0) (6C_3(\mu) + 2C_4(\mu) + 69C_5(\mu) + 32C_6(\mu)) \\
& + \frac{8}{3} (C_3(\mu) + 10C_5(\mu)), \tag{3.4}
\end{aligned}$$

where

$$h(\mu, s, m_q) = \frac{2}{9} \left(\ln \frac{m_q^2}{\mu^2} - \frac{2}{3} - z \right) - \frac{1}{9} (2 + z) B_0(s, m_q), \tag{3.5}$$

with $z = 4m_q^2/s$, and

$$B_0(s, m_q) = -2 \sqrt{|z-1|} \begin{cases} \arctan \frac{1}{\sqrt{z-1}} & z > 1 \\ \ln \frac{1 + \sqrt{1-z}}{\sqrt{z}} - \frac{i\pi}{2} & z \leq 1 \end{cases}. \tag{3.6}$$

We will not consider two-loop corrections to the matrix elements in this paper and concentrate on the one-loop corrections which have been dealt with in previous works [18–21]. In all these papers a different operator basis was used. To compare those results with ours, one can simply use the formulae given in appendix A.

In [18, 19] the findings of Inami and Lim [22] were used to estimate the Wilson coefficient C_9 from electroweak theory without QCD. It was later pointed out by Fajfer et al. [20] that this leads to a great overestimation of the decay width. We agree with the authors on that point. However, in 2011 Paul et al. [21] argued that those results contain a sign error in the function analogous to our function h defined in (3.5), which would invalidate the main arguments given in [20]. We will therefore try to clarify this point again in a slightly different way than in [20].

Let us first look at the case of B -decays. To obtain the matching condition at the scale M_W at leading order for the Wilson coefficient C_9 , one first has to calculate penguin and box diagrams in full QCD. This calculation has been performed by Inami and Lim [22]. The result contains logarithms of the form $\log(m_t/M_W)$ and $\log(m_c/M_W)$. The u -quark mass is set to zero and the corresponding IR-singularity is regularised dimensionally. Then the corresponding diagrams have to be computed within the effective theory. The Wilson coefficient has to be chosen in such a way that both calculations coincide at the scale M_W , where the matching can be performed at zero momentum transfer. The effective theory calculation thus leads to terms proportional to the h -function in (3.5) at $s = 0$:

$$h(\mu, 0, m_q) = \frac{2}{9} \left(1 + \log \frac{m_q^2}{\mu^2} \right). \tag{3.7}$$

Again, the u -quark mass is set to zero and therefore the corresponding diagram vanishes within dimensional regularisation. As the top-quark does not appear in the effective theory,

the term containing $\log(m_t/M_W)$ can obviously not be reproduced unless it is contained in $C_9(M_W)$. The $\log(m_c/\mu)$ term in (3.7) matches exactly the $\log(m_c/M_W)$ term from the full QCD calculation which leads to a $\log(\mu/M_W)$ term in $C_9(\mu \sim M_W)$, i.e. the explicit logarithms for the light quark masses in the full theory have *the same* sign as in the quark loop function $h(\mu, s, m_q)$. This is what is expected, as $m_c \ll M_W$, and the corresponding contributions are considered long-distance (as compared to the scale M_W) and should be reproduced within the effective theory and not be contained in the Wilson coefficient. In the actual matching calculation one of course sets m_c to zero from the beginning which leads to the same result for $C_9(M_W)$.

In the case of D -decays, the roles of t -, c - and u -quarks are taken over by b -, s - and d -quarks. By the same reasoning as before, this time *all* the quark masses can be set to zero in the matching calculation which immediately leads to vanishing $C_9(M_W)$ due to the unitarity of the CKM-matrix. When Paul et al. [21] state that the logarithms in the Inami-Lim term and in the effective QCD corrections have to have a different sign, it should be clear from the above considerations that this cannot be true. Moreover, the function h has a smooth limit for $m_q \rightarrow 0$ at $s \neq 0$:

$$h(\mu, s, 0) = -\frac{2}{27} \left(2 + 3\pi i - 3 \log \frac{s}{\mu^2} \right). \quad (3.8)$$

If the logarithm in the Inami-Lim term were to cancel the explicit logarithm in the first term in (3.5), the whole contribution would contain a logarithmic divergence for vanishing quark masses at $s \neq 0$.

4 Conclusions and outlook

In this paper we have presented the calculation of Wilson coefficients for the weak effective Hamiltonian relevant for rare semileptonic decays of D -mesons at NNLL order which is required to perform an analysis of those decays at first order in the strong coupling α_s . The calculation is very similar to the analogous one for B -meson decays. The main difference arises through the necessity to perform a two-step matching, as one has to cross the b -quark threshold while evolving the renormalisation scale from the high scale M_W down to the charm-mass scale. The corresponding anomalous dimensions and initial conditions at M_W could be taken from the results known in the B -meson sector, with the obvious replacements of quark charges and number of flavours within the effective theory. We tried to clarify some misunderstanding present in the literature concerning the correct matching at the scale M_W .

As mentioned in the introduction, due to the specific CKM and mass structure of charm FCNCs, the short distance contributions within the SM can differ by several orders of magnitude depending on which corrections are taken into account. We have seen that many of the Wilson coefficients are very similar to the ones for b -decays. Only C_9 differs by one order of magnitude and C_{10} is zero. To fully exploit the SM short-distance contributions one of course has to take into account the hadronic matrix elements within the effective theory. This will be done at the same order in the strong coupling α_s in a future publication [23].

	\bar{C}_1	\bar{C}_2	\bar{C}_3	\bar{C}_4	\bar{C}_5	\bar{C}_6
LL	−0.517	1.266	0.010	−0.025	0.007	−0.029
NLL	−0.356	1.157	0.014	−0.042	0.010	−0.045
NNLL	−0.317	1.140	0.013	−0.040	0.009	−0.045

Table 2. “Barred” Wilson coefficients \bar{C}_{1-6} at the scale $\mu = 1.3 \text{ GeV}$ in leading-logarithmic (LL), next-to-leading-logarithmic (NLL) and next-to-next-to-leading-logarithmic (NNLL) order. Input parameters are the same as in table 1.

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A Alternative operator bases

For comparison with previous work we will introduce “barred” coefficients \bar{C}_i (for $i = 1, \dots, 6$), defined by the following linear combinations of the Wilson coefficients C_i [12]:

$$\begin{aligned}
 \bar{C}_1 &= \frac{1}{2} C_1, \\
 \bar{C}_2 &= C_2 - \frac{1}{6} C_1, \\
 \bar{C}_3 &= C_3 - \frac{1}{6} C_4 + 16 C_5 - \frac{8}{3} C_6, \\
 \bar{C}_4 &= \frac{1}{2} C_4 + 8 C_6, \\
 \bar{C}_5 &= C_3 - \frac{1}{6} C_4 + 4 C_5 - \frac{2}{3} C_6, \\
 \bar{C}_6 &= \frac{1}{2} C_4 + 2 C_6.
 \end{aligned} \tag{A.1}$$

The linear combinations are chosen such that the \bar{C}_i coincide *at leading logarithmic order* with the Wilson coefficients in the standard basis [4]. Numerical values for the coefficients are listed in table 2. These definitions hold to all orders in perturbation theory. The “barred” coefficients are related to those defined in [4] by [7]

$$\bar{C}_i = C_i^{\text{BBL}} + \frac{\alpha_s}{4\pi} T_{ij} C_j^{\text{BBL}} + O(\alpha_s^2), \tag{A.2}$$

where

$$T = \begin{pmatrix} \frac{7}{3} & 2 & 0 & 0 & 0 & 0 \\ 1 & -\frac{2}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{178}{27} & -\frac{4}{9} & \frac{160}{27} & \frac{13}{9} \\ 0 & 0 & \frac{34}{9} & \frac{20}{3} & -\frac{16}{9} & -\frac{13}{3} \\ 0 & 0 & \frac{164}{27} & \frac{23}{9} & -\frac{146}{27} & -\frac{32}{9} \\ 0 & 0 & -\frac{20}{9} & -\frac{23}{3} & \frac{2}{9} & \frac{16}{3} \end{pmatrix}. \tag{A.3}$$

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